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Pulse Broadening in Multimode Optical Fibers

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Closed-form expressions are obtained for the impulse response of graded-index fibers whose relative permittivity is a homogeneous function of the two transverse coordinates x, y, and for the impulse width in graded-index fibers whose profile departs slightly, but otherwise arbitrarily, from a square law. The inhomogeneous dispersion of the material is taken into account. Pulse broadening can be reduced by a factor of 12 from the value obtained for square-law fibers. Simple expressions are found for the acceptance of highly oversized fibers.

I. INTRODUCTION

Light-emitting diodes supply their optical power in a time and space incoherent form. The line width is typically of the order of 200 Å, and the radiation is approximately lambertian with an emissive area of the order of 50×50 μm. Time and space incoherent optical pulses can be transmitted by oversized optical fibers. However, optical pulses propagating in such fibers tend to broaden as they travel. This is in part due to the nonzero line width of the source and the dispersion $(d^2k/d\omega^2)$ of the fiber material. The other cause of pulse broadening is associated with the fact that the time of flight of a pulse along a ray depends on the ray trajectory. Pulses traveling along axial rays usually go faster than pulses traveling along rays of large amplitude. Because both types of rays are excited by spatially incoherent sources, the difference in axial group velocity causes a broadening of the input pulse. In the main text of this paper, we assume that the carrier is

monochromatic and that the spatial distribution of the rays is time-invariant. This is the case, for instance, when the source is an injection laser that oscillates simultaneously on many transverse modes. The difference in frequency between these various transverse modes can usually be neglected.

It was first pointed out by Kompfner[1] that pulse broadening in step-index fibers could be drastically reduced by introducing ray equalizers at various locations along the fiber. The role of ray equalizers is to exchange fast and slow rays. A possible implementation of this idea is shown in Fig. 1 together with the calculated impulse response for uncorrected and corrected step-index fibers.[2] Because natural mode mixing appears to be very small in the most recently made optical fibers, ray converters may be practical. They have not been experimented with, however, and we shall therefore restrict ourselves to uniform, uncorrected fibers.

Important results concerning the broadening of spatially incoherent optical pulses in graded-index fibers have already been reported. In Refs. 3 to 9, the difference in group velocity between the various modes that can propagate in step-index and graded-index fibers bas

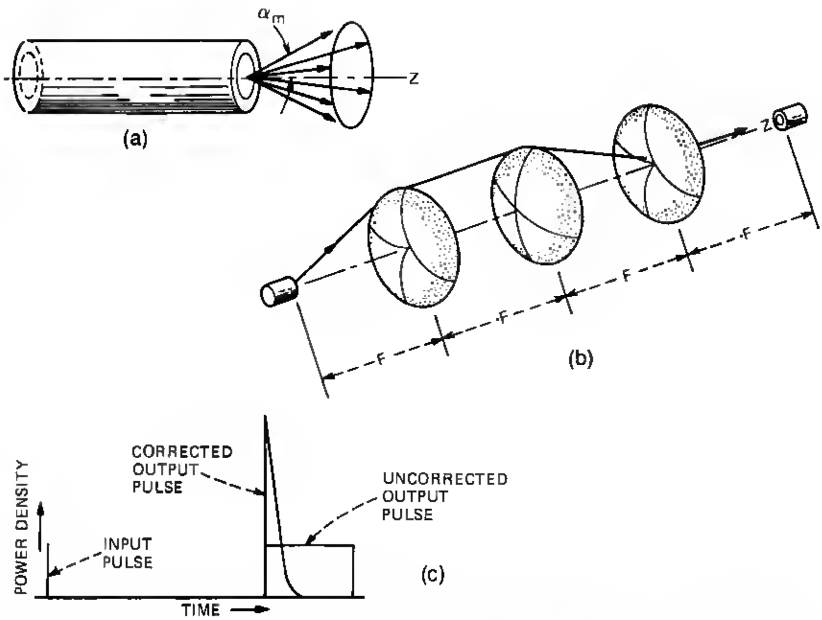

Fig. 1—Ray converter that minimizes pulse broadening in step-index fibers. (a) Angular spread of a step-index fiber. (b) Optical arrangement with confocal lenses. The first and last lenses are unconventional. (c) Calculated impulse response for uncorrected [$P(t) = 1$ within the pulse] and corrected step-index fibers (from Ref. 2).

been evaluated. The impulse response is obtained by adding the contribution of each mode, under the assumption that all modes are equally excited by the source. The calculation of the group velocities can be simplified with the help of the W.K.B. approximation (see, in particular, Ref. 4).

Let us now describe an alternative ray-optics method. The time of flight of a pulse along a ray is first evaluated according to the laws of geometrical optics. A ray can be defined by the point x, y where it intersects the input plane of the fiber (plane $z = 0$), and by the transverse components, k_x, k_y of the wave vector $\mathbf{k}$. $\mathbf{k}$ is, by definition, directed along the ray and has magnitude $(2\pi/\lambda_0)n$, where λ_0 denotes the free-space wavelength and n the refractive index of the fiber material, usually a function of x and y. Thus, the time of flight of a pulse (at a fixed carrier frequency) is, in general, a function of the four parameters x, y, k_x, k_y. These four parameters can be considered the components of a four-vector $\mathbf{p}$, in the so-called phase space. The impulse response is subsequently obtained by assuming that the density of rays is equal to $(2\pi)^{-2}$ in the phase space. In other words, we assume that the number of rays whose points of intersection with the input plane are between x, $x + dx$ and y, $y + dy$, and whose direction is defined by values of k_x, k_y between k_x, $k_x + dk_x$ and k_y, $k_y + dk_y$, is equal to $dxdy\, dk_xdk_y/(2\pi)^2$. The total power transmitted is the acceptance (or number of modes) of the fiber. This is the power transmitted for a source of luminance unity (see, for example, Ref. 10).

The approach used in Refs. 11 to 13 is based on the conventional ray equations. We have shown in Refs. 14 and 15 that it brings a considerable simplification to write the ray equations in the Hamiltonian form. The relationship between the ray-optics method and the W.K.B. method becomes more obvious with the Hamiltonian form. It can be shown that the W.K.B. method and the ray-optics method are essentially identical.[14]

An important difference, however, should be noted. In the W.K.B. method, modes whose axial wave number k_z is less than the free wave number k_s in the surrounding medium (or cladding) are assumed to leak out so rapidly that they can be ignored. On that basis, the acceptance of a step-index round fiber with radius a, for example, is found to be

$$N^2 = (k^2 - k_s^2)a^2/2 \equiv V^2/2.$$

The radiation loss of leaky modes can be small in the case of highly oversized fibers, however, as was pointed out by Snyder.[16] The ray-optics condition is distinctly different: Only those rays are ignored whose tangential component of the wave vector at the core-cladding

interface $[(k_z^2 + k_\varphi^2)^{\frac{1}{2}}$, where k_φ denotes the azimuthal wave number$]$ is less than the free wave number k_s in the surrounding medium. According to ray optics, the acceptance of a step-index fiber is $N^2 = V^2$ instead of $V^2/2$. The influence of the slightly leaky rays on the impulse response of fibers has not been observed. This is perhaps because high-order modes are more sensitive to irregularities than low-order modes. Slightly leaky rays may become important when highly oversized fibers of good quality are fabricated. This is even more so for graded-index (e.g., near-square-law) fibers, because the field decays exponentially beyond the caustic line, which bounds the ray trajectories.

In most previous works, the effect of inhomogeneous dispersion* on quasi-monochromatic pulse broadening was neglected. This effect, however, was taken into account for square-law and linear-law graded-index fibers in Appendix B of Ref. 14, and by Gambling and Matsuhara[9] for circularly symmetric modes in square-law fibers perturbed by an r^4 term. The result for arbitrary small deviations from square-law was given by Arnaud in Ref. 15. Olshansky and Keck[9] first pointed out that inhomogeneous dispersion is of great practical importance, at least for fibers doped with TiO_2. Dispersion for the promising GeO_2 doped fibers is not known at the time of this writing. The variation of the loss of that material as a function of doping is likewise unknown. If we consider further that the sources used in pulse broadening experiments are not fully characterized in terms of their distribution in phase space, it appears that a precise comparison between theory and experiment is difficult at the moment. We shall therefore restrict ourselves to the theoretical evaluation of pulse broadening.

II. GENERAL RESULTS

The derivations of the general results given in this section appear in Appendix A. They follow in a straightforward manner from the Hamilton equations for pulse trajectories in space-time.

Fibers are most often characterized by a refractive-index profile: $n(x, y, \omega)$. However, the quantity that enters directly in the expressions for pulse broadening is the square of the wave number $k^2(x, y, \omega) \equiv (2\pi/\lambda_0)^2 n^2(x, y, \omega)$, where λ_0 denotes the wavelength in free space. We shall therefore deal directly with $k^2(x, y, \omega)$.

Let $x(z)$, $y(z)$ denote a ray trajectory. Assuming that the fiber is time-invariant and uniform and that the material is isotropic, we ob-

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* Inhomogeneous dispersion refers to the spatial variations of the ratio of the local phase to group velocities in the material. This parameter should not be confused with the parameter $d^2k/d\omega^2$, usually refered to as "material dispersion." The latter is important only for broadband carriers.

tain from the ray equations the following differential equation (see Appendix A):

$$\tfrac{1}{2}k_z^2 d^2(X + Y)/dz^2 = k^2 - k_z^2 + X\partial k^2/\partial X + Y\partial k^2/\partial Y, \qquad (1)$$

where we have set, for convenience, $X \equiv x^2(z)$, $Y \equiv y^2(z)$. The quantity k_z in (1) denotes the axial (z) component of the wave vector $\mathbf{k}$ and is a constant of motion. In other words, k_z remains the same along any given ray. In a wave theory, k_z corresponds to the propagation constant of a mode (sometimes denoted β). Note that, in spite of the fact that we are using the language of wave optics, the theory given in this paper is based strictly on ray optics, except when we impose the condition $k_z > k_s$ to make contact with previous results.

It follows from the space-time ray equations that the ratio of the time of flight of a pulse along a ray to the corresponding time on axis is (see Appendix A)

$$t = (k_0/k_z)\langle\partial k^2/\partial\omega^2\rangle/(dk_0^2/d\omega^2), \qquad (2a)$$

where $k_0 \equiv k(0, 0, \omega)$. The sign $\langle\ \rangle$ denotes an average over a ray period. For any function $a(x, y, \omega)$, we have defined

$$\langle a(x, y, \omega)\rangle \equiv Z^{-1}\int_0^Z a[x(z), y(z), \omega]dz, \qquad (2b)$$

where $x(z)$, $y(z)$ denotes a particular ray trajectory and Z the ray period. If the ray trajectory is not periodic, $\langle a\rangle$ should be understood as the limit of the right-hand side of (2b) when $Z \to \infty$. In the special case where the inhomogeneous dispersion of the material can be neglected, k is proportional to ω and, consequently, $\partial k^2/\partial\omega^2 = k^2/\omega^2$. Thus, (2a) reduces to

$$t = \langle k^2\rangle/(k_0 k_z). \qquad (2c)$$

Finally, if the source of rays has a distribution $f(\mathbf{p})$ in the phase space $\mathbf{p} \equiv \{x, y, k_x, k_y\}$, the response of the fiber to an input $P'(t)$ is (see Appendix A)

$$P(t) = \int P'[t - t(\mathbf{p})]f(\mathbf{p})T(\mathbf{p})(d\mathbf{p}). \qquad (3)$$

The quantity $T(\mathbf{p})$ is the transmission of a ray (usually $T < 1$), and $(d\mathbf{p}) \equiv dxdy\, dk_x dk_y$. In the special case of a uniform lambertian source of luminance unity, we have $f(\mathbf{p}) = 1/(2\pi)^2$. For simplicity, we can assume that $T(\mathbf{p})$ is unity when the point x, y is within the core cross section and the components k_x, k_y of $\mathbf{p}$ are within some area to be defined later for specific examples and zero outside that area. All the subsequent results follow from (1), (2), and (3).

III. IMPULSE RESPONSE WHEN $k^2(x, y) - k_0^2$ IS A HOMOGENEOUS FUNCTION OF x AND y

Let the differential equation (1) be averaged over a ray period. The left-hand side of (1) vanishes because $d(X + Y)/dz$ assumes the same values at the ends of the integration interval. (In this integration, k_z can be considered a constant.) Thus, we have

$$\langle k^2 - k_z^2 + X\partial k^2/\partial X + Y\partial k^2/\partial Y\rangle = 0. \tag{4}$$

Let us further assume that $h(X, Y) \equiv k^2(X, Y) - k_0^2$ is a homogeneous function of degree κ in $X \equiv x^2$ and $Y \equiv y^2$. This means that, for any λ,

$$h(\lambda X, \lambda Y) = \lambda^\kappa h(X, Y). \tag{5}$$

If we differentiate (5) with respect to λ and set $\lambda = 1$, we obtain

$$X\partial h/\partial X + Y\partial h/\partial Y = \kappa h(X, Y). \tag{6}$$

Thus, going back to $k^2(x, y, \omega)$,

$$X\partial k^2/\partial X + Y\partial k^2/\partial Y = \kappa(k^2 - k_0^2). \tag{7}$$

In that case, a simple and general expression for the relative delay in the absence of material dispersion is readily obtained from (2c), (4), and (7),

$$t = [(k_z/k_0) + \kappa(k_0/k_z)]/(1 + \kappa). \tag{8}$$

The relative delay t is plotted in Fig. 2 as a function of k_z/k_0 with κ as a parameter. This result is applicable, for example, to the index profile

$$k^2(x, y) = k_0^2 - \alpha|x| - \beta|y|, \tag{9}$$

where α and β denote constants. In that example, $\kappa = \frac{1}{2}$. Note that the fiber described by (9) is not circularly symmetric, even if $\alpha = \beta$. Examples of circularly symmetric fibers that satisfy (5) will be given in the next section.

In almost any z-invariant focusing system, any initial distribution eventually reaches a steady state. This steady state in general differs from the initial distribution. A lambertian distribution f = constant, however, remains lambertian because it is a (trivial) solution of the Liouville equation (see Appendix A). Note that the distribution f in (3) represents a ray (or pulse) density. If the medium introduces a nonuniform attenuation on the rays, the power density $T(\mathbf{p})f(\mathbf{p})$ in phase space needs to be distinguished from the distribution f.

A fiber is usually surrounded by a homogeneous material, called the cladding, with wave number k_s. For fibers that are not highly

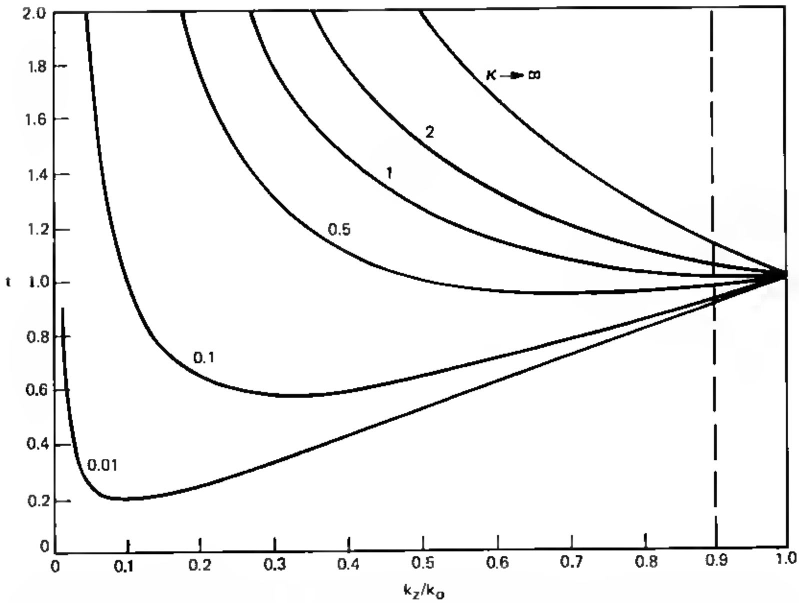

Fig. 2—Relative time of flight in a fiber where $k^2(x, y) - k_0^2$ is a homogeneous function of degree κ in x and y [$k_0 \equiv k(0, 0)$]. For most fibers, k_z/k_0 is close to unity.

overmoded, the transmission law

$$T(x, y, k_x, k_y) = \begin{cases} 1 & \text{if} \quad k_z > k_s \\ 0 & \text{if} \quad k_z \leqq k_s \end{cases} \tag{10}$$

is often acceptable. Equation (10) says that rays whose axial wave number is less than the free wave number in the surrounding medium are leaking sufficiently rapidly to be ignored. The distribution f of the lambertian source is set equal to $1/(2\pi)^2$ so that the luminance is unity. In that case, the total power transmitted is the acceptance of the fiber. The relative time of flight is, within the present assumptions, solely a function of k_z. The upper and lower bounds on k_z are $k(x, y)$ and k_s, respectively. It remains to express the volume element $dk_x dk_y dx dy$ in (3) as a function of dk_z, dx, dy. For given x, y, a constant value of k_z corresponds to a circle of radius squared $k^2(x, y) - k_z^2$ in the k_x, k_y space because $k_x^2 + k_y^2 = k^2(x, y) - k_z^2$. Thus,

$$dk_x dk_y = \pi dk_z^2. \tag{11}$$

Let us evaluate the acceptance of the fiber. The light acceptance of any optical system is, as we have seen, the volume in phase space of the accepted rays divided by $(2\pi)^2$. It is also equal to the effective number of modes that the system can transmit. If we integrate $P(t)$

from $t = -\infty$ to $t = +\infty$ in (3), the integral over P' in the integrand is unity, and we obtain

$$N^2 = (1/4\pi) \iint [k^2(x, y) - k_s^2] dx dy, \tag{12}$$

where we have used (10) and (11). Thus, $4\pi N^2$ is the volume enclosed by the profile: $k^2(x, y)$. For a step-index fiber of any shape with cross-section area A, for example, we have from (12)

$$N^2 = (A/4\pi)(k^2 - k_s^2). \tag{13}$$

This expression should be multiplied by 2 to take into account the two states of polarization.

The pulse transformation in (3) becomes, using (11),

$$P(t) = (1/4\pi) \int dx dy \int_{k_s^2}^{k^2(x,y)} P'[t - t(k_z)] dk_z^2. \tag{14}$$

Let the input pulse $P'(t)$ be a symbolic δ function (e.g., a rectangular pulse of width Δt and height Δt^{-1} in the limit $\Delta t \rightarrow 0$). The output pulse in (14) becomes

$$P(t) = (1/4\pi) |dk_z^2/dt| A(k_z), \qquad k_z > k_s, \tag{15}$$

where $|dk_z^2/dt|$ denotes the absolute value of dk_z^2/dt and $A(k_z)$ denotes the cross-section area that satisfies $k(x, y) > k_z$. k_z can be expressed as a function of the delay t by inverting the relation $t(k_z)$ given earlier. We obtain, from (8),

$$dk_z'^2/dt = 2(1 + \kappa) k_z' [1 - (\kappa/k_z'^2)], \tag{16}$$

where

$$k_z' = k_z/k_0 = (1 + \kappa)t/2 \pm \{[(1 + \kappa)t/2]^2 - \kappa\}^{\frac{1}{2}}. \tag{17}$$

If $\kappa > 1$, there is only one value of k_z' between $k_s' \equiv k_s/k_0$ and 1, for any k_s'. If

$$k_s'^2 < \kappa < 1, \tag{18}$$

there are two values of k_z' that need be considered. Their contributions to P should be added. If $\kappa < k_s'^2$, there is again only one relevant value of k_z'.

Let us consider as an example a (noncircularly symmetric) square-law medium

$$k^2(x, y) = k_0^2(1 - \Omega_x^2 x^2 - \Omega_y^2 y^2), \tag{19}$$

where Ω_x, Ω_y denote arbitrary constants. $2\pi/\Omega_x$ and $2\pi/\Omega_y$, for small x, y, are the periods of ray oscillation in the xz and yz planes, respectively. The area $A(k_z)$ defined earlier is the interior of an ellipse

$$A(k_z) = \pi(1 - k_z'^2)/\Omega_x \Omega_y. \tag{20}$$

The impulse response is obtained from (15) and (16) with $\kappa = 1$, and (20),

$$P(t) = \begin{cases} k_0^2 k_z'^3/(\Omega_x\Omega_y), & k_z > k_s \\ 0 \quad , & k_z \leqq k_s, \end{cases} \tag{21}$$

where, from (17), $k_z' = t - (t^2 - 1)^{\frac{1}{2}}$. Because, in most fibers, k_z remains close to k_0, the variation of k_z' can be neglected, and the pulse response is almost rectangular.

For a step-index fiber, the area A is the area of the core cross section, and $t = k_0/k_z$. Thus, the impulse response of a step-index fiber with cross-section area A is simply

$$P(t) = k_0^2 A/2\pi t^3, \qquad 1 < t < k_0/k_s. \tag{22}$$

Because, in most fibers, t remains close to unity, the pulse response is almost rectangular.[2] The pulse width, however, is considerably larger than for square-law fibers, as we shall see in more detail later.

IV. CIRCULARLY SYMMETRIC FIBERS WITH $k^2 - k_0^2$ A POWER OF THE RADIUS

Let the wave-number profile be of the form

$$k^2(R, \omega) = k_0^2(\omega) - k_\kappa^2(\omega)R^\kappa, \tag{23}$$

where $R \equiv X + Y \equiv r^2$ denotes the square of the radius. The relative time of flight is, substituting (23) in (2a),

$$\begin{aligned} t &= (k_0/k_z)\langle \partial k^2/\partial\omega^2\rangle/(dk_0^2/d\omega^2) \\ &= (k_0/k_z)(1 - \epsilon_\kappa D_\kappa\langle R^\kappa\rangle), \end{aligned} \tag{24}$$

where we have defined

$$\epsilon_\kappa \equiv k_\kappa^2/k_0^2 \tag{25}$$

$$D_\kappa \equiv k_0^2(dk_\kappa^2/d\omega^2)/k_\kappa^2(dk_0^2/d\omega^2). \tag{26}$$

D_κ is a dispersion factor equal to unity in the absence of dispersion. Thus, we need to evaluate $\langle R^\kappa\rangle$. It is interesting that we can do that without solving the ray equations. The quantity $\langle R^\kappa\rangle$ is, of course, independent of dispersion, so we may omit the ω arguments.

For circularly symmetric fibers, (1) can be written

$$\tfrac{1}{2}k_z^2 d^2R/dz^2 = d(k^2R)/dR - k_z^2. \tag{27}$$

Averaging (27) over a ray period, we obtain

$$k_z^2 = \langle d(k^2R)/dR\rangle. \tag{28}$$

We have also, directly from (23),

$$\langle k^2\rangle = k_0^2 - k_\kappa^2\langle R^\kappa\rangle \tag{29a}$$

and, from (28) and (23),

$$\langle k^2 \rangle = (k_z^2 + \kappa k_0^2)/(1 + \kappa). \tag{29b}$$

Equating the two expressions (29a) and (29b) for $\langle k^2 \rangle$, we obtain

$$\epsilon_\kappa \langle R^\kappa \rangle = (1 - k_z'^2)/(1 + \kappa), \tag{30}$$

where $k_z' \equiv k_z/k_0$. Thus, substituting $\langle R^\kappa \rangle$ from (30) into (24), the relative time of flight is

$$t = k_z'^{-1} - D_\kappa(k_z'^{-1} - k_z')/(1 + \kappa). \tag{31}$$

In applications, we need k_z' as a function of t. Solving (31) for k_z' and setting $D_\kappa' \equiv D_\kappa/(1 + \kappa)$, we obtain

$$k_z' = (t/2D_\kappa') \pm [(t/2D_\kappa')^2 + 1 - D_\kappa'^{-1}]^{\frac{1}{2}}. \tag{32}$$

By differentiating (32), we further obtain

$$dk_z'^2/dt = 2k_z'[D_\kappa' - (1 - D_\kappa')/k_z'^2]^{-1}. \tag{33}$$

To obtain explicitly the impulse response in (15), we need the area $A(k_z)$ defined by $k_z < k(R)$. For $k(R)$ in (23), this area is

$$A(k_z) = \pi R(k_z) = \pi[(1 - k_z'^2)/\epsilon_\kappa]^{1/\kappa}. \tag{34}$$

If ϵ_κ were kept a constant as the parameter κ varies, the core radius a, defined by $k(a) = k_s$, would vary. Thus, it is preferable to express ϵ_κ as a function of the core radius a. We have

$$\epsilon_\kappa^{1/\kappa} = (1 - k_s'^2)^{1/\kappa}/a^2, \tag{35}$$

where $k_s' \equiv k_s/k_0$. The impulse response is finally obtained from (15), (33), (34), and (35);

$$P(t) = (k_0^2 a^2/2)k_z'[(1 - k_z'^2)/(1 - k_s'^2)]^{1/\kappa}/[D_\kappa' - (1 - D_\kappa')k_z'^{-2}]. \tag{36}$$

The possibly doubled value k_z' is expressed as a function of t by (32). Thus, a closed-form expression has been obtained for the impulse response of a fiber with $k^2 - k_0^2$ a power of r, that takes inhomogeneous dispersion into account.

In the absence of dispersion, we have $D_\kappa' = 1/(1 + \kappa)$, and (36) reduces to

$$P(t) = (k_0^2 a^2/2)k_z'[(1 - k_z'^2)/(1 - k_s^2)]^{1/\kappa}(1 + \kappa)/(1 - \kappa k_z'^{-2}). \tag{37}$$

As indicated in the previous section, there are in general two values of k_z' that contribute to P. Note that the shape of the impulse response does not depend on the core radius a.

The impulse response $P(t)$ in (37) is shown in Figs. 3 and 4 for various values of the parameter κ. These curves are essentially the

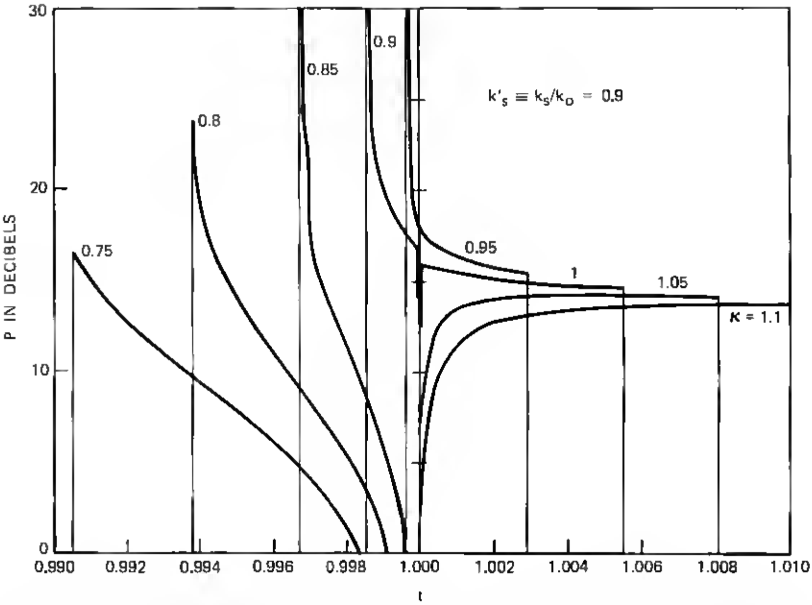

Fig. 3—Impulse response of a circularly symmetric fiber with $k^2(r) = k_0^2 - k_\kappa^2 r^{2\kappa}$ for a lambertian source and various values of κ. The optimum impulse response is for $\kappa \approx k_s' = 0.9$.

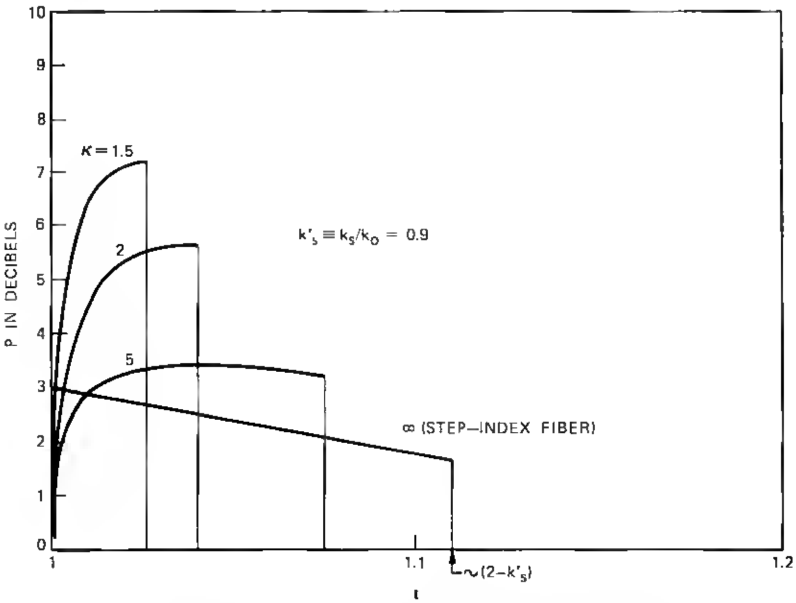

Fig. 4—Continuation of Fig. 3 for larger values of κ.

same as those shown in Ref. 4. Figures 3 and 4, however, are much more detailed. We have assumed that $k_s/k_0 = 0.9$, that is, $\Delta n/n = 10\%$, a rather large value. For $\kappa = 1$ (square-law fiber), the pulse width τ is 0.0054. For example, if $n = 1.45$ and the fiber length is 1 km, the pulse width is 26 ns. For $\kappa = 0.9$, however, the corresponding pulse width is only 7 ns. We find, in agreement with Ref. 4, that the minimum pulse width occurs when $\kappa = k_s'$. For a step-index fiber ($\kappa \rightarrow \infty$), the pulse width would be as large as 630 ns. Note the following detailed features on the curves in Figs. 3 and 4. For $(0.9)^2 < \kappa < 1$, the response starts from infinity because of the minimum in the $t(k_z)$ curve. For $\kappa = 0.85$, P drops suddenly for $t \approx 0.998$. This is because, at that time, the smaller of the two k_z' becomes less than 0.9, and is rejected. For $\kappa = 0.95$, the response crosses the $t = 1$ axis.

Figure 4 is applicable to larger values of κ. We note that, for a very large κ (step-index fiber), the response is almost rectangular. The slow decay in power shown in Fig. 4 would be almost negligible for small $\Delta n/n$.

The effect of inhomogeneous dispersion is shown in Fig. 5. The parameter κ is kept equal to 0.9 (this is the optimum value in the absence of inhomogeneous dispersion), but D_κ is made to vary in the

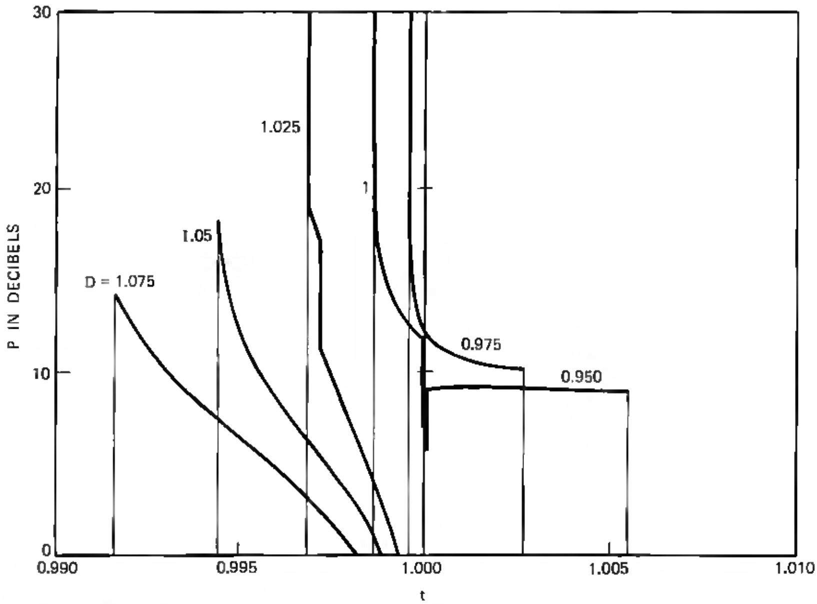

Fig. 5—Impulse response for a fiber with $k^2(r) = k_0^2 + k_\kappa^2(r^2)^{0.9}$ for various values of the parameter D that expresses the inhomogeneous dispersion of the material. $D = 1$ corresponds to the absence of dispersion. $D \neq 1$ merely introduces a shift in the optimum value of κ.

neighborhood of unity. These curves have a striking resemblance to those in Fig. 3. This means that the effect of inhomogeneous dispersion merely consists in shifting the optimum value of κ. The impulse response remains essentially the same, at least for $\kappa \approx 1$.

The total pulse power is the acceptance of the fiber, a function of κ. The acceptance is, in the present case,

$$\begin{aligned} N^2 &= \int_{-\infty}^{+\infty} P(t)dt = (\tfrac{1}{4}) \int_0^a [k^2(R) - k_s^2]dR \\ &= (\tfrac{1}{4})[(k_0^2 - k_s^2)a^2 - k_\kappa^2 a^2 (a^2)^\kappa/(\kappa + 1)] \\ &= [\kappa/4(\kappa + 1)](k_0^2 - k_s^2)a^2. \end{aligned} \tag{38}$$

The coefficient in the last expression in (38) is $\frac{1}{4}$ for step-index fibers ($\kappa \rightarrow \infty$) and $\frac{1}{8}$ for square-law fibers. The acceptance given in (38) should be multiplied by 2 to account for the two states of polarization. The same rule applies to all the expressions given in this paper. It is more difficult to obtain the ray-optics acceptance of fibers. The result is derived in Appendix B.

In the next section, we consider fibers whose profile departs slightly, but otherwise arbitrarily, from a square law.

V. NEAR-SQUARE-LAW FIBERS

Let us rewrite the differential equation (1) for circularly symmetric fibers

$$\tfrac{1}{2}k_z^2 d^2R/dz^2 = d(k^2R)/dR - k_z^2. \tag{39}$$

For square-law fibers

$$k^2(R) = k_0^2 - k_1^2 R, \tag{40}$$

the solution of (39) is

$$R(z) = R_0 + (R_0^2 - l_z^2/k_1^2)^{\frac{1}{2}} \cos (2\Omega z/k_z'), \tag{41}$$

where

$$R_0 \equiv \tfrac{1}{2}(k_0^2 - k_z^2)/k_1^2 = \tfrac{1}{2}(1 - k_z'^2)/\Omega^2 \tag{42}$$

and $\Omega \equiv k_1/k_0$. We have introduced in (41) the axial component of the angular momentum (or Bouguer invariant)

$$l_z = xk_y - yk_x, \tag{43}$$

which is the second constant of motion. Let us set

$$\theta \equiv (l_z/k_1 R_M)^2, \tag{44}$$

where R_M denotes the maximum radius squared. Note that, for meridional rays, $\theta = 0$ and, for helical rays, $\theta = 1$. Equation (41) can be written in the convenient form

$$R = \tfrac{1}{2}R_M(1 + \theta) + \tfrac{1}{2}R_M(1 - \theta) \cos (2\Omega z/k_z'). \tag{45}$$

For later use let us evaluate $\langle R^n \rangle$, the average of R^n over a ray period. Using the binomial expansion and the result

$$\langle \cos^m \rangle = m!2^{-m}[(m/2)!]^{-2} \tag{46}$$

for m even and 0 for m odd, we obtain

$$\langle R^n \rangle = n!2^{-n}R_M^n \sum_{m=0,2\cdots}^{n} \frac{(1+\theta)^{n-m}(1-\theta)^m}{2^m(n-m)![(m/2)!]^2}. \tag{47}$$

In particular,

$$\langle R^2 \rangle = R_M^2(3\theta^2 + 2\theta + 3)/8 \tag{48a}$$

$$\langle R^3 \rangle = R_M^3(1+\theta)(5\theta^2 - 2\theta + 5)/16 \tag{48b}$$

$$\langle R^4 \rangle = R_M^4(35\theta^4 + 20\theta^3 + 18\theta^2 + 20\theta + 35)/128. \tag{48c}$$

Let us now show that a closed-form expression can be obtained for the times of flight in fibers whose permittivity profiles depart slightly from a square law. Inhomogeneous dispersion is taken into account. Let the profile be of the form

$$k^2(R) = k_0^2 - k_1^2 R + \sum_{n=2}^{N} k_n^2 R^n. \tag{49}$$

We assume that $\epsilon_n R^{n-1}$, $n \geqq 2$, is of the order $\epsilon \ll 1$, where $\epsilon_n \equiv k_n^2/k_0^2$.

Substituting (49) in (2a), we obtain (with $\Omega^2 \equiv \epsilon_1 \equiv k_1^2/k_0^2$)

$$t = k_z'^{-1}(1 - D_1\Omega^2\langle R \rangle + \sum_{n=2}^{N} \epsilon_n D_n \langle R^n \rangle), \tag{50}$$

where we have defined inhomogeneous dispersion factors

$$D_n = (k_0^2 dk_n^2/d\omega^2)/(k_n^2 dk_0^2/d\omega^2). \tag{51}$$

The D_n are unity in the absence of inhomogeneous dispersion. Because the perturbation is small, $\langle R^n \rangle$ in the sum (50) can be replaced by the expression (47) applicable to square-law fibers. This approximation is not permissible, however, for the term $\langle R \rangle$ in (50) because this term is not small. We need an exact expression for $\langle R \rangle$. We proceed as in the previous section. We first observe that, for k^2 in (49),

$$d(k^2R)/dR = 2k^2 - k_0^2 + \sum_{n=2}^{N} (n-1)k_n^2 R^n. \tag{52}$$

Integrating (39) over a ray period, the left-hand side vanishes and, using (52), we obtain an expression for $\langle k^2 \rangle$ that does not involve $\langle R \rangle$

$$\langle k^2 \rangle = \tfrac{1}{2}(k_z^2 + k_0^2) + \tfrac{1}{2}\sum_{n=2}^{N} (1-n)k_n^2\langle R^n \rangle. \tag{53}$$

We also have, directly from (44),

$$\langle k^2 \rangle = k_0^2 - k_1^2\langle R\rangle + \sum_{n=2}^{N} k_n^2\langle R^n\rangle. \tag{54}$$

Thus, by comparing (53) and (54),

$$k_1^2\langle R\rangle = \tfrac{1}{2}(k_0^2 - k_z^2) + \tfrac{1}{2}\sum_{n=2}^{N}(n+1)k_n^2\langle R^n\rangle. \tag{55}$$

Substituting this expression for $\langle R\rangle$ in (50), we obtain the relative time of flight for circularly symmetric near-square-law fibers

$$t = k_z'^{-1}\{1 - \tfrac{1}{2}(1 - k_z'^2)D_1 + \sum_{n=2}^{N}[D_n - \tfrac{1}{2}(n+1)D_1]\epsilon_n\langle R^n\rangle\}. \tag{56}$$

Alternatively, t can be expressed in terms of the azimuthal and radial mode numbers. The result is given in Appendix C.

In the absence of inhomogeneous dispersion, (56) reduces to

$$t = \tfrac{1}{2}k_z'^{-1}[k_z'^2 + 1 + \sum_{n=2}^{N}(1-n)\epsilon_n\langle R^n\rangle]. \tag{57}$$

Limiting ourselves to an r^4 correction to the square-law profile, $\epsilon_3 = \epsilon_4 = \cdots = 0$, and setting $\epsilon_2 \equiv \epsilon$, (57) becomes, using (48),

$$\begin{aligned} t &= \tfrac{1}{2}[1 - \rho_M(1+\theta)]^{-\frac{1}{2}}[2 - \rho_M(1+\theta) - \epsilon\rho_M(3\theta^2 + 2\theta + 3)/8] \\ &\approx 1 + \rho_M^2[(2 - 3\epsilon) + (4 - 2\epsilon)\theta + (2 - 3\epsilon)\theta^2]/16 + 0(\rho_M^3) \\ \rho_M &\equiv \Omega^2 R_M. \end{aligned} \tag{58}$$

The first two terms in (58) give sufficient accuracy when $\rho_M \lesssim 0.01$, that is, when the total relative change in refractive index $\Delta n/n \approx \rho_a/2$ is less than 0.005 ($\rho_a \equiv \Omega^2 a^2$).

The total pulse width τ is the maximum variation of t for $0 < \theta < 1$ and $0 < \rho < \rho_a$. For the square-law fiber [$\epsilon = 0$ in (58)], we obtain

$$\tau = \rho_a^2/2 \qquad \text{(ray optics)}. \tag{59}$$

It should be noted that, in defining τ in (59), we have specified that the maximum radius of the ray be less than a for any θ. This condition is different from the condition used earlier that k_z be larger than k_s. The ray-optics condition $\rho_M < \rho_a$ is applicable to highly oversized fibers.

If we now consider the expression in (58) with a correction term in r^4, we find that $t = 1$ for meridional rays ($\theta = 0$) when $\epsilon_2 = \frac{2}{3}$ in agreement with Ref. 17, where it is shown that all the rays have exactly the same optical length when $k^2(x) = [\cosh(x)]^{-2} \approx 1 - x^2$

$+(\frac{2}{3})x^4 + \cdots$. We also find that $t = 1$ for helical rays ($\theta = 1$) when $\epsilon_2 = 1$, in agreement with Ref. 3, where it is shown that helical rays have exactly the same optical length when $k^2(r) = (1 + r^2)^{-1} \approx 1 - r^2 + r^4 + \cdots$. By considering all rays whose maximum radius is less than a, we find that the minimum τ is obtained for $\epsilon_2 = 0.91$. In that case, $\tau = 0.046\rho_a^2$. The improvement compared with square-law media is therefore as large as 11. If we had imposed instead the wave-optics condition $k_z > k_s$, the vertical scale in Fig. 6 would be divided by $(1 + \theta)^2$. For $\epsilon_2 = 0$, for example, the wave-optics pulse width is $\rho_a^2/8$ instead of $\rho_a^2/2$ as in (59). With the wave-optics limit, the optimum value of ϵ_2 turns out to be $\frac{2}{3}$ instead of 0.91. The improvement over the square-law case is only 4, instead of 11.

Let us now consider the effect of r^6 terms. Figure 7 shows the variation of the pulse width τ, defined as the maximum variation of t for any $0 < \theta < 1$ and any $0 < \rho_M < 0.002$, as a function of ϵ_2 for various values of ϵ_3. The effect of ϵ_3 is essentially to shift the optimum value of ϵ_2 to lower values. The reduction in pulse width is rather modest. Nevertheless, a small improvement is obtained, compared to the case

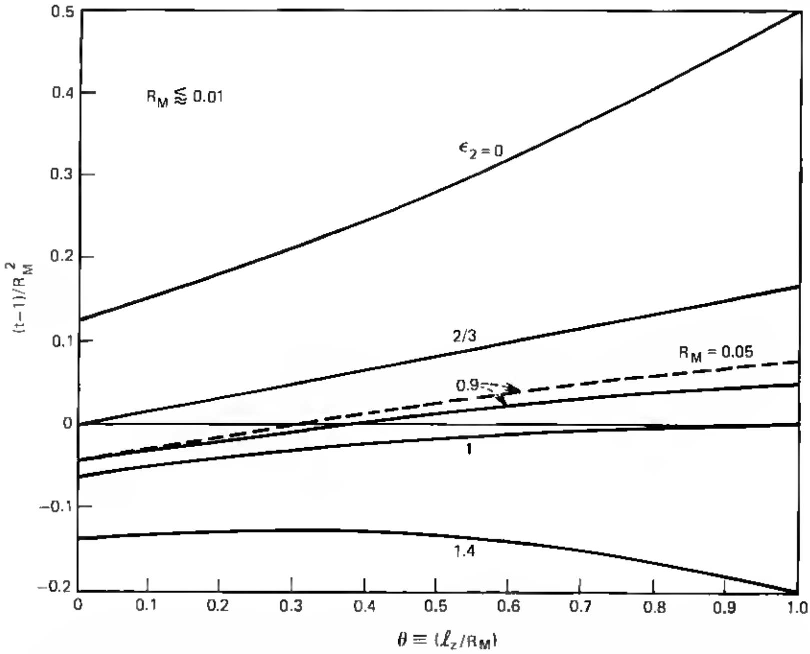

Fig. 6—Variation of the normalized time of flight for a fiber with $k^2(r) = k_0^2 - k_1^2r^2 + \epsilon_2(k_1^4/k_0^2)r^4$ in the absence of material dispersion for various values of the parameter ϵ_2. $\theta = 0$ corresponds to meridional rays and $\theta = 1$ to helical rays. ϵ_2 has been redefined to be dimensionless.

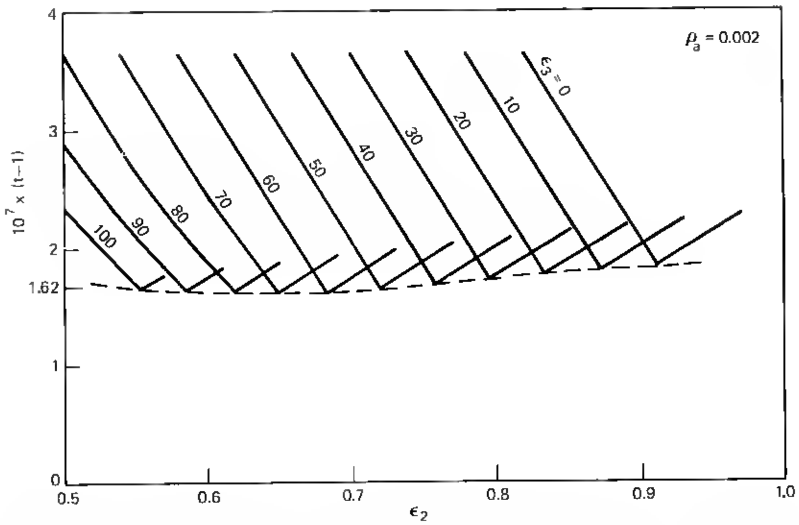

Fig. 7—Variation of the width of the impulse response with ϵ_2 for various values of ϵ_3 for a fiber with $k^2(r) = k_0^2 - k_1^2 r^2 + \epsilon_2(k_1^4/k_0^2)r^4 + \epsilon_3(k_1^6/k_0^4)r^6$.

where $\epsilon_3 = 0$ when

$$k_0^{-2}k^2(r) = 1 - \rho + 0.615\rho^2 + 70\rho^3, \qquad \rho_a = 0.002 \qquad (60)$$
$$\rho \equiv \Omega^2 r^2.$$

We give only the result when the departures from a square-law profile are not circularly symmetric. The free wave number in the fiber is now in the form

$$k^2(x, y, \omega) = k_0^2(\omega) - k_1^2(\omega)R + \sum_{n=1}^{N}\sum_{l=0}^{n} k_{nl}^2(\omega)X^l Y^{n-l}, \qquad (61)$$

where, as before, $X \equiv x^2$, $Y \equiv y^2$, $R \equiv x^2 + y^2 \equiv r^2$. The ratio, t, of the time of flight along a ray to the corresponding time on-axis is found to be

$$t = k_z'^{-1}\{1 - \tfrac{1}{2}(1 - k_z'^2)D_1 + \sum_{n=1}^{N}\sum_{l=0}^{n}[D_{nl} - \tfrac{1}{2}(n+1)D_1]$$
$$\times \epsilon_{nl}\langle X^l Y^{n-l}\rangle\}, \qquad (62)$$

where $\epsilon_{nl} \equiv k_{nl}^2/k_0^2$ and D_{nl} is defined as D_n in (51) with k_n replaced by k_{nl}. Let us assume that it is permissible to use the sinusoidal rays of the square-law medium to evaluate the quantity $\langle X^l Y^{n-l}\rangle$. Because the average over one cycle of the product of powers of sinusoidal functions is known, the relative delay t can be written in closed form.

Let the ray trajectory be written

$$x(z) = x_0 \cos(\alpha z + \phi_x) \tag{63a}$$

$$y(z) = y_0 \cos(\alpha z + \phi_y). \tag{63b}$$

The coefficient α does not enter in the final result and is henceforth omitted. We evaluate

$$\begin{aligned}\langle X^l Y^{n-l} \rangle &\equiv \langle x^{2l} y^{2(n-l)} \rangle \\ &= x_0^{2l} y_0^{2(n-l)} \langle [\cos(z + \phi_x)]^{2l} [\cos(z + \phi_y)]^{2(n-l)} \rangle. \end{aligned} \tag{64}$$

It can be shown that[18]

$$\begin{aligned}&\langle [\cos(z + \phi_x)]^{2l} [\cos(z + \phi_y)]^{2(n-l)} \rangle \\ &\quad = 2^{-2n} \left\{ 2 \sum_{s=1}^{n-l} \binom{2(n-l)}{n-l-s} \binom{2l}{l-s} \cos[2s(\phi_x - \phi_y)] \right. \\ &\qquad \left. + \binom{2(n-l)}{n-l} \binom{2l}{l} \right\}, \end{aligned} \tag{65}$$

where

$$\binom{a}{b} \equiv \frac{a!}{(a-b)!b!}. \tag{66}$$

Thus, given a ray trajectory, defined by the parameters x_0, y_0, ϕ_x, and ϕ_y (or, equivalently, by the values of x, y, k_x, and k_y at the input of the fiber), we can evaluate in closed form the quantity $\langle X^l Y^{n-l} \rangle$ that enters in formula (62) for the relative time of flight, from (64) to (66).

The above calculation is incomplete for the following reasons. When the power law $n^2(r)$ of a fiber departs from the exact square law, projected ray trajectories in the (xy) transverse plane are *precessing* ellipses.* That is, the principal axes of the near-elliptical trajectories slowly rotate as a function of z. This precession is unimportant for circularly symmetric fibers. For noncircularly symmetric fibers, however, the ellipse precession introduces an averaging effect. Furthermore, the noncircularly symmetric components of $n^2(r, \varphi)$ change the *eccentricity* of the precessing ellipse. The axial component k_z of the wave vector remains a constant, but the axial component l_z of the angular momentum varies. Finally, in real fibers, slow (adiabatic) changes of the refractive index law along the fiber axis are likely to occur that must be taken into account. The twists of the fiber axis must also be taken into account. Thus, a realistic assessment of the effect of small noncircularly symmetric departures of the index law

* It is well known in mechanics that the only $r^{2\kappa}$ potentials (potential $U \sim n^2$) that give closed trajectories are the harmonic potential $U(r) \sim n^2(r) = 1 - r^2$ and the Newton potential $U(r) \sim n^2(r) = 1/r$.

from square law on pulse broadening requires a deeper and more intricate analysis than the one given in the present section. However, the result in (62) and (65) can be used as a basis for more complete analyses.

VI. CONCLUSION

From a rather straightforward application of the Hamilton ray equations, we have obtained closed-form expressions for the pulse width in graded-index fibers when $k^2(x, y) - k_0^2$ is a homogeneous function of x and y, and for fibers whose profile departs slightly, but otherwise arbitrarily, from a square law. Inhomogeneous dispersion was taken into account. The expressions obtained are exact. The small angle (or weakly guiding) approximation need not be made. We have also given simple expressions for the wave optics and ray optics acceptance of weakly guiding graded-index fibers.

The algebraic results given should prove more accurate and require much less computer time than the straightforward numerical integration of time along ray trajectories. We have carried the perturbation only to first order in the small parameter ϵ. To obtain more accurate results, up to order ϵ^2, we need a more accurate expression of the ray trajectory. This expression can be obtained, for example, by the method of strained coordinates.[19] These more accurate expressions are probably not needed, however, in most practical cases.

VII. ACKNOWLEDGMENT

The author expresses his thanks to E. A. J. Marcatili for useful comments.

APPENDIX A

The Hamilton Equations of Ray Optics

The Hamiltonian form of the ray equations are well known in mechanics and wave dynamics,[20] and they have also been used frequently in optics (e.g., Refs. 14, 15, 17, 21, and 22). However, their simplicity and power is not always appreciated. The physical difficulty is that it is not always recognized that ray momenta and wave vectors (or photon momenta) are identical concepts. On the other hand, ray momenta (proportional to the wave vectors) need be carefully distinguished from mass-carrying momenta (proportional to the group velocities).[23] On the mathematical side, we need distinguish a function such as $k_z(x, y)$ and the value k_z assumed by that function. We must also be aware that da/dz denotes a total derivative, that is, in the present context, the variation of the quantity a along some given ray.

If a is a known function of x and y, and $x = x(z)$, $y = y(z)$ denote a known ray trajectory, then $da/dz = (\partial a/\partial x)(dx/dz) + (\partial a/\partial y)(dy/dz)$ can be evaluated explicitly as a function of z. Here again, an arbitrary point in space x, y should not be confused with a specific ray trajectory $x = x(z)$, $y = y(z)$. Unfortunately, it is not possible to go into more details here. An excellent reference is Lighthill's paper.[20] A comparison between the W.K.B. method and the Hamilton equations is given in Ref. 14.

Let $\mathbf{X} \equiv (x, y, z, ict)$ denote a point in space-time (t is time) and $\mathbf{K} \equiv (k_x, k_y, k_z, i\omega/c)$ denote the four-wave vector, with ω the angular frequency. An arbitrary medium is characterized by a function of $\mathbf{K}$ and $\mathbf{X}$ that we denote

$$H(\mathbf{K}, \mathbf{X}) = 0. \tag{67}$$

The Hamilton equations for light pulses $\mathbf{X}(\sigma)$, $\mathbf{K}(\sigma)$ are

$$d\mathbf{X}/d\sigma = \partial H/\partial \mathbf{K} \tag{68a}$$

$$d\mathbf{K}/d\sigma = -\partial H/\partial \mathbf{X}, \tag{68b}$$

where σ denotes an arbitrary parameter.

Equations (68a) and (68b) can be considered the basic postulates of geometrical optics. From a wave-optics point of view, (68a) follows from the requirement that the wave lengths and periods of the waves that constitute a wave packet be the same in the direction of a ray. Equation (68b) follows from (67), (68a), and the fact that $\mathbf{K}$ is the gradient of an eikonal function. Thus, in wave optics, the Hamilton equations (68) are derived from first principles and need not be postulated.

Let ξ denote a point in phase space $(k_x, k_y, \omega, x, y, t)$ at the input plane, and ξ' a point in phase space at the output plane. The optical system maps the input phase space into the output phase space, that is,

$$\xi = \xi(\xi'). \tag{69}$$

It follows from (67) and (68) that the Jacobian of the transformation (69) is unity, a result often used in photometry. Equivalently, we can say that the determinant of paraxial ray matrices is unity or that the ray density in phase space is a constant of motion (Liouville theorem). These three statements are obviously equivalent, provided the rays are not reflected.

A source of light that is time and space incoherent is described by a distribution $S(\xi)$ in phase space. Each small volume in phase space can be pictured as an optical pulse, provided the dimensions of the volume are larger than unity. More precisely, this picture requires that $\Delta\omega\Delta t \gg 1$, $\Delta k_x \Delta x \gg 1$, and $\Delta k_y \Delta y \gg 1$. The detailed structure

of the pulse is ignored in ray optics. Only the motion of the center is considered.

The transmission T_1 of an optical pulse through the optical system is a presumably known function of ξ that we denote as

$$T_1 = T_1(\xi). \tag{70}$$

For lossy optical systems, $T_1 < 1$. Because the Jacobian of the transformation $\xi \to \xi'$ is unity, the output distribution is simply

$$S'(\xi') = S(\xi)T_1(\xi). \tag{71}$$

The power emitted by the source and the power that can be collected at the output of the optical system are obtained by integrating S (or S') over all variables, except t (or t'). Thus,

$$P(t) = \int S(\xi)(d\zeta) \tag{72a}$$

$$P'(t') = \int S'(\xi')(d\zeta'), \tag{72b}$$

where $\zeta \equiv (k_x, k_y, \omega, x, y)$ and ζ' is similarly defined. The terms $(d\zeta)$ and $(d\zeta')$ denote elementary volumes in ζ and ζ' spaces, respectively. The response of the detector could be described by a function $D(\xi')$. For simplicity, we do not take the detector response into consideration. All subsequent results follow in a rather straightforward manner from the above results, through a succession of approximations.

Let us assume that the properties of the fiber do not vary with time. This means that the Hamiltonian in (67), the transmission T_1, the mapping $\xi \to \xi'$, and the pulse delay do not depend on time. In particular,

$$t' = t + t_1(\zeta). \tag{73}$$

Sources that are t-separable, on the other hand, have the property that

$$S(\xi) = P(t)F(\zeta). \tag{74}$$

That is, the distribution in ζ-space does not vary with time. For a hot tungsten wire whose temperature varies with time, the spatial phase-space distribution is almost lambertian at all times, but the frequency spectrum (approximately given by the Plank law of radiation) varies with time. Thus, (74) is not applicable to that source. For consistency with (72), we assume that $F(\zeta)$ is normalized to unity.

For most sources, we can further assume that

$$F(\zeta) = \Omega(\omega)f(\mathbf{p}), \tag{75}$$

where $\mathbf{p} \equiv (k_x, k_y, x, y)$ denotes a point in spatial phase-space. That is, we assume that the spatial distribution does not depend on what part of the frequency spectrum we are considering. Both Ω and f are assumed normalized to unity. This ensures that F is normalized to unity. When the spectral width of the source is small (e.g., less than 1 percent, as is the case for light-emitting diodes) and the fiber material absorption does not exhibit sharp resonances in that band, we can assume that

$$T_1(\boldsymbol{\zeta}) = T_0(\omega)T(\mathbf{p}) \tag{76}$$

and

$$t_1(\boldsymbol{\zeta}) = t_0(\omega) + t(\mathbf{p}). \tag{77}$$

For definiteness, we assume that the maximum value of $T_0(\omega)$ is unity, and we define $t_0(\omega)$ as the delay experienced by axial pulses. We evaluate in the main text $t(\mathbf{p})/t_0$ at a fixed angular frequency.

The pulse response is obtained from (71) to (77),

$$\begin{aligned} P'(t') &= \int P[t' - t_0(\omega) - t(\mathbf{p})]\Omega(\omega)T_0(\omega)f(\mathbf{p})T(\mathbf{p})(d\mathbf{p})d\omega \\ &= \int P''[t' - t_0(\omega)]\Omega(\omega)T_0(\omega)d\omega, \end{aligned} \tag{78}$$

where

$$P''(t'') = \int P[t'' - t(\mathbf{p})]f(\mathbf{p})T(\mathbf{p})(d\mathbf{p}). \tag{79}$$

In writing (78) we have used the fact that the Jacobian of the transformation $\boldsymbol{\xi} \rightarrow \boldsymbol{\xi}'$ is unity and that $d\omega = d\omega'$. The pulse response is the convolution of the pulse response in (79), which we may call the quasi-monochromatic pulse response, and the spectral width of the source. In most cases, $T_0(\omega)$ is a constant. For injection lasers, the quasi-monochromatic pulse response is the most important contribution.

In what follows, we assume that the fiber is uniform and long compared with the period of ray oscillation and therefore approximately z-invariant. Let the Hamiltonian in (67) be written

$$H = k_z - k_z(k_x, k_y, \omega, x, y) = 0. \tag{80}$$

The Hamilton equations (68) are

$$dx/dz = -\partial k_z/\partial k_x \tag{81a}$$

$$dy/dz = -\partial k_z/\partial k_y \tag{81b}$$

$$dt/dz = \partial k_z/\partial \omega \tag{81c}$$

$$dk_x/dz = \partial k_z/\partial x \tag{81d}$$

$$dk_y/dz = \partial k_z/\partial y. \tag{81e}$$

Let us assume further that the medium is isotropic, that is,

$$k_z^2 = k^2(\omega, x, y) - k_x^2 - k_y^2. \tag{82}$$

Thus, (81a) to (81e) are

$$dx/dz = k_x/k_z \tag{83a}$$

$$dy/dz = k_y/k_z \tag{83b}$$

$$dt/dz = (\partial k^2/\partial\omega)/2k_z \tag{83c}$$

$$dk_x/dz = (\partial k^2/\partial x)/2k_z \tag{83d}$$

$$dk_y/dz = (\partial k^2/\partial y)/2k_z. \tag{83e}$$

According to (83c), the time of flight of a pulse along a ray for a period (period $\equiv Z$) is obtained by integrating $(\partial k^2/\partial\omega)/2k_z$ from $z = 0$ to $z = Z$. If $k_0(\omega) \equiv k(\omega, 0, 0)$ denotes the wave number on axis, the time of flight of a pulse along the z axis is similarly obtained by integrating $(\partial k_0^2/\partial\omega)/2k_0$. Thus, the ratio of the time of flight of a pulse along a ray to the corresponding time on axis is

$$t = (k_0/k_z)\langle\partial k^2/\partial\omega^2\rangle/(dk_0^2/d\omega^2), \tag{84}$$

where $\langle\ \rangle$ denotes an average over a ray period. If the trajectory is not periodic, $\langle\ \rangle$ is understood as a limit for $z \to \infty$. When k is proportional to ω, (84) reduces to

$$t = \langle k^2\rangle/k_0k_z. \tag{85}$$

Let us now observe that, from (83a), (83b), (83d), and (83e),

$$\tfrac{1}{2}k_z^2 d^2(X + Y)/dz^2 = k^2 - k_z^2 + X\partial k^2/\partial X + Y\partial k^2/\partial Y, \tag{86}$$

where $X \equiv x^2$, $Y \equiv y^2$. This is easily verified by carrying out the differentiations. Equations (86), (84), and (79) (with a slightly different notation) are those used in the main text.

APPENDIX B

Acceptance of Highly Oversized Fibers

The acceptance, or effective number of modes transmitted by the optical system, is the volume of the accepted rays in phase space divided by $(2\pi)^2$. We have said earlier that, if the fiber is very long, all leaky rays are eliminated and the acceptance is simply the volume enclosed by the profile $k^2(x, y)$ divided by 4π. If the fiber is highly oversized, however, many leaky rays ($k_z < k_s$) are not significantly attenuated.[16] We need then consider the ray-optics condition that the tangential (rather than axial) component of the wave vector be larger than k_s at the core-cladding interface. The ray-optics acceptance is

now evaluated for circularly symmetric fibers. We specify that

$$k_z^2 + k_\varphi^2 > k_s^2, \qquad \text{at} \quad r = a, \tag{87a}$$

where k_φ denotes the azimuthal wave number at the interface. We also have the condition

$$k_z^2 > 0, \tag{87b}$$

which is not implied by (87a). In this appendix, we restrict ourselves to small differences in refractive index, in which case condition (87b) can be ignored. Because of the conservation of l_z (the axial component of the angular momentum), we have

$$rk_y = ak_\varphi \tag{88}$$

for a ray with $x = r$, $y = 0$, k_x, k_y, at the input plane, that can reach the interface $r = a$. Thus, condition (87a) is

$$k^2(r) - k_x^2 - k_y^2 + (r^2/a^2)k_y^2 > k_s^2. \tag{89}$$

Equation (89) defines an area in the k_x, k_y plane bounded by an ellipse. We have to make sure, however, that rays outside that area do in fact reach the interface. This is not necessarily the case. The maximum ray radius r_M is defined implicitly by

$$k_x^2 + (1 - r^2/r_M^2)k_y^2 = k^2(r) - k^2(r_M), \tag{90}$$

where r_M is the largest real number that satisfies (90). (The initial radius r is considered a constant in the present discussion.) Equation (90) shows that the k_x, k_y that correspond to r_M are contained in an ellipse with semi-axes squared $k_{x0}^2 = k^2(r) - k^2(r_M)$ and $k_{y0}^2 = [k^2(r) - k^2(r_M)]/(1 - r^2/r_M^2)$, respectively. If $k^2(r)$ is never increasing, we are sure that k_{x0} keeps increasing as r_M increases from r to a. We do not have any such assurance for k_{y0}, however. When r_M reaches a, there may be acceptable values of k_x, k_y that are located outside the ellipse defined above. For each profile, we need therefore verify that $k_{y0}^2(r_M)$ never exceeds $k_{y0}^2(a)$. We easily verify that this is the case for square-law fibers, because

$$k_{y0}^2 = k_1^2(r_M^2 - r^2)/(1 - r^2/r_M^2) = k_1^2 r_M^2 \tag{91}$$

increases with r_M for any r.

Thus, for square-law fibers at least, we can proceed with the calculation of the area of the ellipse defined by (89). This area is

$$\pi[k^2(r) - k_s^2](1 - r^2/a^2)^{-\frac{1}{2}}. \tag{92}$$

Substituting this result in the general expression for the acceptance

factor, we obtain

$$N^2 = (\tfrac{1}{4}) \int_0^{a^2} [k^2(r) - k_s^2](1 - r^2/a^2)^{-\frac{1}{2}} dr^2. \tag{93}$$

This expression simplifies if we introduce the variable $u \equiv (1 - r^2/a^2)^{\frac{1}{2}}$. Equation (93) becomes

$$N^2 = (a^2/2) \int_0^1 [k^2(u) - k_s^2] du. \tag{94}$$

Thus, the ray-optics acceptance of most circularly symmetric fibers is half the area enclosed by the curve $k^2(u)a^2$. For a step-index fiber, we obtain from (94)

$$N^2 = (k_0^2 - k_s^2)a^2/2 \quad \text{(step-index, ray optics).} \tag{95}$$

This is twice the wave-optics acceptance. Thus, for step-index fibers, tbe slightly leaky rays carry half the power. Our result agrees with that in Ref. 16 for weakly guiding fibers. For a square-law fiber, with $k(a) = k_s$, we obtain

$$N^2 = (k_0^2 - k_s^2)a^2/6 \quad \text{(square-law, ray optics).} \tag{96}$$

In square-law fibers, 25 percent of the total power is carried by slightly leaky rays.*

APPENDIX C

Impulse response width of near-square law fibers

When the source distribution is lambertian, all propagating modes are equally excited. It is convenient in that case to express tbe relative time of flight t for near-square-law fibers given in (56) as a function of the mode numbers (azimuthal number: $\mu = \cdots -2, -1, 0, 1, 2 \cdots$ and radial number: $\alpha = 0, 1, 2 \cdots$) rather than k_z and l_z. This can be done by quantizing the ray trajectories. [If tbe W.K.B. method is used, it is essential to first remove tbe singularity of the Helmboltz equation at $r = 0$. This is achieved by changing the independent variable from r to $\log(r)$.] One easily finds that the axial component of the ray angular momentum l_z is equal to μ. Furtbermore, we can use for k_z the well-known expression applicable to square-law media. The result (56) is written below as a function of α, μ, for the reader's convenience. We have

$$t(\alpha, \mu) = (1 - B)^{-\frac{1}{2}} \left(1 - \tfrac{1}{2} B D_1 + \sum_{\gamma=2}^{\infty} F_\gamma N_\gamma \right), \tag{97}$$

* This is in agreement with a recent result by D. N. Payne.

where

$$B \equiv 2gK_1^{1/2}/K_0$$
$$g \equiv 2\alpha + |\mu| + 1; \quad |\mu| \equiv \text{abs. val. } (\mu)$$
$$F_\gamma \equiv \gamma!2^{-\gamma}[D_\gamma - \tfrac{1}{2}(\gamma+1)D_1]K_\gamma/(K_0K_1^{\gamma/2})$$
$$N_\gamma \equiv (2g)^\gamma \sum_{m=0,2}^{\gamma} \{2^m(\gamma - m)![(m/2)!]^2\}^{-1}[1 - (\mu/g)^2]^{m/2}. \quad (98)$$

The parameters $K_\gamma \equiv k_\gamma^2$, $\gamma = 0, 1 \cdots$ and D_γ, $\gamma = 1, 2 \cdots$ are obtained from the square of the wave number: $K(R) \equiv k^2(R) \equiv (\omega/c)^2 n^2(R)$ of the fiber as a function of $R \equiv r^2$, measured at the nominal wavelength λ_0 and at a slightly different wavelength, λ_0', expanded in power series of R as follows

$$\begin{aligned} K(R) &= K_0 - K_1R + K_2R^2 + \cdots \quad (\lambda_0) \\ K'(R) &= K_0' - K_1'R + K_2'R^2 + \cdots \quad (\lambda_0'). \end{aligned} \quad (99)$$

The D_γ are obtained from (99)

$$D_\gamma = K_0(K_\gamma' - K_\gamma)/K_\gamma(K_0' - K_0). \quad (100)$$

If we can neglect the power in the leaky modes, the mode numhers α, μ are restricted by the condition $k_z > k_s$, that is,

$$B < 1 - K_s/K_0 \approx 2\Delta n/n, \quad (101)$$

where $K_s \equiv k_s^2$ is the square of the cladding wave number. The root-mean-square impulse response width is defined as

$$\sigma = 5{,}000[\langle t^2\rangle - \langle t\rangle^2]^{\frac{1}{2}} \text{ ns/km}, \quad (102)$$

where $\langle\ \rangle$ denotes an average taken over all the modes permitted by (101). Thus, it is a straightforward matter to evaluate from our expression in (56) the root-mean-square width of the impulse response of any circularly symmetric near-square law fiher, provided the wave-number profile can be measured with sufficient accuracy at two wavelengths.

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